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**BOOLEAN ALGEBRA APPLIED TO DETERMINATION  
OF UNIVERSAL SET OF KNOWLEDGE STATES****Kikumi K.Tatsuoka**

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Boolean Algebra

Boolean Algebra Applied to Determination of  
Universal Set of Knowledge States

Kikumi K. Tatsuoka

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# ABSTRACT

Diagnosing cognitive errors possessed by examinees can be considered as a pattern classification problem which is designed to classify a sequential input of stimuli into one of several predetermined groups. The sequential inputs in our context are item responses and the predetermined groups are various states of knowledge resulting from misconceptions or different degrees of incomplete knowledge in a domain. In this study, the foundations of a combinatorial algorithm that will provide the universal set of states of knowledge will be introduced. Each state of knowledge is represented by a list of "can/cannot" cognitive tasks and processes (called cognitively relevant attributes or latent variables) which are usually unobservable. A Boolean descriptive function will be introduced as a mapping between the attribute space spanned by latent attribute variables and the item response space spanned by item score variables. The Boolean descriptive function plays the role of uncovering the unobservable content of a black box. Once all the possible classes are retrieved explicitly and expressed by a set of ideal item response patterns which are described by a "can/cannot" list of latent attributes, the notion of bug distributions and statistical pattern classification techniques will enable us to diagnose students' states of knowledge accurately. Moreover, investigations on algebraic properties of these logically-derived-ideal-response patterns will provide an insight into the structures of the test and dataset.

## Introduction

A typical pattern classification problem is to classify a sequential input of stimuli into one of several predetermined groups. The predetermined groups are considered, in our context, as latent classes which represent various states of knowledge and capabilities, and the stimuli are item response patterns. Tatsuoka (1983, 1985) introduced a cognitive error diagnostic model (called rule space) in which a student's response pattern to the items is classified into one of the predetermined latent classes. Each latent class consists of binary patterns that deviate from a given ideal response pattern by various numbers of slippages. Tatsuoka & Tatsuoka (1987) introduced the slippage probabilities and showed that such a class of response patterns follows a statistical distribution (called a Bug distribution). The ideal response pattern is the outcome of the perfectly consistent execution of some erroneous rule of operation or the response pattern corresponding to some state of knowledge and capabilities without errors of measurement. An error analysis or a task analysis usually provides a list of erroneous rules of operations and/or various sources of misconceptions which are regarded as latent classes in this paper. However, it is important to have a systematic method for obtaining an appropriate list of ideal response patterns automatically. The method must be

applicable to any domain of interest. In this paper, such a method and the theoretical foundation of the method are introduced. The theoretical foundation is built upon algebraic relations between observable item patterns and latent score patterns of various cognitive tasks. Boolean Lattice theory is applied to develop the theoretical foundation of a test and data structure.

#### An Incidence Matrix and Binary Scoring

Suppose that the underlying characteristics of a domain of interest are well identified and involvement relationships between the latent attribute variables  $A_k$ ,  $k=1, \dots, K$  (also called cognitively relevant attributes) and items are coded by a binary matrix. The matrix is called an incidence matrix. Let the incidence matrix be a  $K \times n$  matrix  $Q$  where  $K$  is the number of attributes and  $n$  is the number of items. The row vectors of  $Q$ ,  $A_k$ ,  $k = 1, \dots, K$  indicate which items involve the attribute  $A_k$ . Let latent variable  $Y_k$  be the score of attribute task  $A_k$ ; that is,  $Y_k = 1$  if attribute  $A_k$  is correctly performed and  $Y_k = 0$ , otherwise (if  $A_k$  is not a task, and the word "score" is not suitable, then  $Y_k = 1$  could signify "applicable", "belonging to" or any "affirmative adjective"). Let  $X_j$  be a score variable of item  $j$  and assume that  $X_j$  takes the value 1 for the correct answer

and 0 for wrong answers. The relationship between latent-score pattern  $y = (Y_1, \dots, Y_K)$  and the observable item score  $X_j$  is given by Equation (1):

$$X_j = \prod_{k=1}^K Y_k^{Q_{kj}} \quad j=1, \dots, n \quad (1)$$

This equation implies that a response to item  $j$  will be correct if and only if latent scores  $Y_k$  of attribute  $A_k$  for  $Q_{kj} = 1$  are all equal to 1. If any one of such latent scores is zero, then the item score  $X_j$  becomes zero. Needless to say, the meaning of  $Q_{kj} = 0$  and  $Y_k = 0$  should not be confused because  $Q_{kj}$  is an involvement index of attribute  $A_k$  to item  $j$  while  $Y_k$  is the score of attribute task  $A_k$ . The latent score pattern for item  $j$  shall be expressed by

$$z_j = (Y_1 Q_{1j}, Y_2 Q_{2j}, \dots, Y_K Q_{Kj}) \quad (2)$$

where  $Z_{kj} = Y_k Q_{kj}$  does not exist when  $Q_{kj} = 0$ .

Further let us assume the conditional independence of latent variables  $Y_k$  ( $k = 1, \dots, K$ ) and manifest variables  $X_j$  ( $j = 1, \dots, n$ ) for each performance level  $\theta$ . Let  $t$  be the total score of a latent score pattern  $y$  where we assume a special case,  $Q_{kj} = 1$  for

$k = 1, \dots, K$ , and  $p_k$  be the probability of attribute  $A_k$  to be performed correctly,

$$t = \sum_{k=1}^K Y_k \quad (3)$$

Then the random variable  $t$  follows a binomial distribution if the attribute probabilities  $p_k$  are the same for all  $k$ , and a compound binomial distribution if the probabilities are different.

$$\text{Prob}(t|\theta) = \left\{ \sum_{m=0}^t \left\{ \sum_{\sum Y_k = m} \prod_{k=1}^K p_k^{Y_k} (1-p_k)^{1-Y_k} \right\} \right\} \quad (4)$$

The probability of getting the total score of  $K$ , or equivalently the pattern of all ones  $(1, 1, \dots, 1)$  is given by the last term of equation (4).

$$\text{Prob}(t = K|\theta) = \prod p_k \quad (5)$$

Let  $s_j$  be the total score of a latent pattern  $z_j$  for item  $j$ , then the relationships parallel to equations (4) and (5) for the variables  $s_j$ ,  $z_j$  and the probabilities  $p_k$  are given by equations (6) and (7), respectively.

$$\text{Prob}(s_j|\theta) = \left\{ \sum_{n=0}^{s_j} \left\{ \sum_{\Sigma Z_{jk} = s_j} \prod_{k=1}^K p_k^{z_{jk}} (1-p_k)^{1-z_{jk}} \right\} \right\} \quad (6)$$

The probability of getting a particular pattern  $z_j = 1$  is given by equation (7),

$$\text{Prob} \left( s_j = \sum_{k=1}^K Q_{kj} | \theta \right) = \prod_{k=1}^K p_k \quad (7)$$

When item score  $X_j$  is not binary and the response to item  $j$  is scored by taking some partial knowledge into account, then the above discussion needs to be modified.

#### An Incidence Matrix And Partial Credit Scoring

The elements of a latent pattern  $z_j = (Z_{1j}, Z_{2j}, \dots, Z_{Kj})$  of item  $j$  can be replaced by integers or real numbers. Each element  $Z_{kj}$  can be the number of attributes which an examinee answered correctly or the weighted sum of the number of attributes answered correctly. That is:

$$X_j = \sum_{k=1}^K Q_{kj} = \sum_{k=1}^K W_{kj} Z_{kj}, \quad (8)$$

where  $Z_{kj} = 1$  if and only if attribute  $A_k$  is involved in item  $j$  and an examinee performed  $A_k$  correctly.

When  $W_{kj}$  is equal to 1 for  $k = 1, \dots, K$ , then  $X_j$  becomes simply the number of correct attributes. The larger the  $X_j$  value is, the higher the level of performance is. Thus, graded response or partial credit models (Samejima, F., 1969; Masters, G.N., 1982) can be applied. However,  $Z_{kj}$ ,  $k \in \{Q_{kj} = 1\}$  are usually not observable. If a multiple-choice item is constructed so as to have various subsets of scores of  $Z_{kj}$  for the alternatives, then it is possible to apply graded, partial credit or Polychotomous models. The partial credit model is formulated for situations in which ordered response choices are free to vary in number and difficulty from item to item. The restriction of the model is that tests are constructed with an ordered response format. Polychotomous models (Bock, R.D., 1972) do not require the ordered response format and are applicable to multinomial response categories.

When the weights are not 1, then the  $W_{kj}$ 's indicate that the quality of  $A_k$  varies over the attributes. Some attributes are more difficult while others are less so. It is well known that there are  $\binom{s}{s_j}$  ways to get the total score of  $s_j$  from  $s$  different attributes. Some combinations are cognitively more important than

others. It will provide us with useful information for constructing a good item pool for constructed response items or selection of distractors in multiple choice items.

### Lattice and Boolean Algebra

In the previous section, the attributes were introduced as the row vectors of the incidence matrix  $Q$  and denoted by vectors  $A_k$ ,  $k=1, \dots, K$ . For example, let us consider the  $3 \times 5$  incidence matrix shown below, where  $i_1, \dots, i_5$  are items and  $A_1$ ,  $A_2$ , and  $A_3$  are attributes:

$$Q = \begin{matrix} & i_1 & i_2 & i_3 & i_4 & i_5 \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{matrix} \quad (9)$$

There are three row vectors,  $A_1 = (1 \ 0 \ 1 \ 0 \ 1)$ ,  $A_2 = (0 \ 1 \ 0 \ 0 \ 1)$  and  $A_3 = (0 \ 1 \ 1 \ 1 \ 1)$ . In other words, attribute  $A_1$  is involved in items 1, 3 and 5, attribute  $A_2$  is in items 2 and 5 and attribute  $A_3$  is in 2, 3, 4 and 5. Therefore, the attributes can also be expressed by a set theoretical notation like  $A_1 = \{1, 3, 5\}$ ,  $A_2 = \{2, 5\}$  and  $A_3 = \{2, 3, 4, 5\}$ . When we discuss the attributes in the context of set theory, the attributes are written in non-boldface capital letters as  $A_1, \dots, A_K$ . If an incidence matrix  $Q$  happens to be the identity matrix of order  $K = n$ , then  $A_k$  contains

a single item, and  $A_k = \{k\}$ .

Let  $L$  be a set of subsets obtained from the set of  $K$  numbers,  $J = \{1, 2, \dots, K\}$ .  $L$  will be a lattice and Boolean algebra. Lattice and Boolean algebra have been discussed in the field of abstract algebra and they have many interesting properties. They have been applied to digital computer systems and proved to be very useful in providing a simple and precise foundation for the analysis of combinatorial switching circuits. These properties will play a crucial role in achieving our goal which is to obtain the universal set of ideal response patterns (or all the possible states of knowledge and capabilities) obtainable from a given incidence matrix. Let us start from the definition of a lattice.

Definition 1 A set of sets  $L$  is said to be a lattice if two binary compositions  $\cup$  and  $\cap$  are defined on its subsets (called elements hereafter) and they satisfy the following relations:

$$1_1 \quad A \cup B = B \cup A, \quad A \cap B = B \cap A$$

$$1_2 \quad (A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C)$$

$$1_3 \quad A \cup A = A, \quad A \cap A = A$$

$$1_4 \quad (A \cup B) \cap A = A, \quad (A \cap B) \cup A = A$$

The above conditions are equivalent to saying that a lattice is a partially ordered set in which any two elements have a least upper bound and a greatest lower bound. The l.u.b. and g.l.b. of any

elements  $A$  and  $B$  in  $L$  are given by the union and intersection,  $A \cup B$ , and  $A \cap B$ , respectively. Similarly,  $(A \cup B) \cup C$  is the l.u.b. of  $A$ ,  $B$ ,  $C$  and  $(A \cap B) \cap C$  is the g.l.b. The order  $\geq$  in  $L$  is defined by Definition 2:

Definition 2 For any pair of elements  $A$  and  $B$  in  $L$ ,  $A \geq B$  if and only if  $A \cup B = A$  or  $A \cap B = B$ .

Definition 2 provides us with an equivalent condition for  $L$  to be a lattice. This order satisfies the asymmetric (if  $A \geq B$  and  $B \geq A$  then  $A = B$ ) and transitivity laws (if  $A \geq B$  and  $B \geq C$  then  $A \geq C$ ), thus  $L$  becomes a partially ordered set. Let us further define  $I$  and  $O$  as follows:

$$I = \bigcup_{k=1}^K A_k \quad \text{and} \quad O = \bigcap_{k=1}^K A_k \quad (10)$$

then  $I$  and  $O$  belong to  $L$ . If the distributive law,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad (11)$$

is satisfied, then  $L$  is called a modular lattice. The modular condition has an alternative definition: if  $A \geq B$  and  $A \cup C = B \cup C$  and  $A \cap C = B \cap C$  for any  $C$  in  $L$ , then  $A = B$ .

The third important operation is complementation.

Definition 3 The complement  $A'$  of  $A$  is defined by  $A' \cup A = I$  and

$$A' \cap A = 0.$$

For example, the lattice of a set of subsets is complemented if the complement of a subset A is the usual set-theoretic complement--that is, the elements of J that do not belong to A.

Definition 4 A Boolean algebra is a lattice with 1 and 0, the distributive law and complementation.

Definition 4 implies that our lattice L is also a Boolean algebra. The most important elementary properties of complements in a Boolean algebra may be stated as follows:

Theorem 1 The complement  $A'$  of any element A of a Boolean Algebra L is uniquely determined. The mapping  $A \rightarrow A'$  is one to one, onto itself. Then the mapping satisfies conditions 1 and 2:

1.  $(A')' = A$
2.  $(A \cup B)' = A' \cap B'$  and  $(A \cap B)' = A' \cup B'$

The proof may be found in Birkoff (1970).

A Boolean algebra becomes a Ring with the two operations + and x where + is the union set of A and B and x is the intersection of A and B.

Definition 5 For A and B in L, the addition + of A and B is defined by  $A + B = A \cup B$  and the product x is defined by  $A \times B = A \cap B$ . Thus L becomes a Ring.

It is obvious that L satisfies commutative laws, associative laws, the identity laws  $A + 0 = A$ ,  $A \times 1 = A$ , and Idempotent law

$A + A = A$  with respect to the new operations  $+$  and  $\times$ .

Distributive law is also satisfied. In summary,

$$1. A + B = B + A, A \times B = B \times A$$

Commutative Laws

$$2. (A + B)' = A' \times B', (A \times B)' = A' + B'$$

Complementation

$$3. (A + B) + C = A + (B + C), (A \times B) \times C = A \times (B \times C)$$

Associative Laws

$$4. A + 0 = 0 + A = A, A \times I = I \times A = A$$

Identity

$$5. A + A = A, A \times A = A$$

Idempotence

$$6. (A + B) \times C = A \times C + B \times C$$

Distributive Law

The relationship between the attribute vectors

$A_k$  ( $k = 1, \dots, K$ ) and the Ring  $L$  just introduced will be clarified.

#### Attribute Response Space and Item Response Space

When an incidence matrix is the identity matrix of order  $K$ , then  $A_k$  will be the unit vector  $e_k = (0, \dots, 1, 0, \dots, 0)$ , whose  $k$ -th element is 1 and the other elements are zero. A Boolean

lattice  $L$  will then consist of a set of attributes where attribute  $A_k$  corresponds one-to-one to item  $k$  or equivalently to  $e_k$ . Therefore,  $L$  can be considered as a set of sets of items, or equivalently as a set of sets of  $e_k$ s. In order to distinguish between these two sets, the set of sets of attributes is denoted by the same notation,  $L$  and the set of sets of items (or sets of  $e_k$ ) is denoted by  $RL$ , in other words Boolean Algebra of Item Response Patterns. Both  $L$  and  $RL$  are  $K$ -dimensional spaces since the incidence matrix is the identity of order  $K$ . If an incidence matrix is not the identity then  $RL_1$  which associates with a non-identity incidence matrix becomes a subspace of  $RL$ . It is very difficult, in practice, to construct an item-pool whose incidence matrix is the identity. Each item in the identity-incidence matrix must contain one and only one attribute. It is very common that an item involves several attributes and two different items usually involve two different sets of attributes. In practice most incidence matrices are usually more complicated than the identity matrix and their columns and rows contain several ones in a variety of cells.

In the earlier example of  $3 \times 5$  matrix,  $A_1 = (1 \ 0 \ 1 \ 0 \ 1)$  corresponds to set  $A_1 = \{1, 3, 5\}$ ;  $A_2 = (0 \ 1 \ 0 \ 0 \ 1)$  to  $A_2 = \{2, 5\}$ ; and  $A_3 = (0 \ 1 \ 1 \ 1 \ 1)$  to  $A_3 = \{2, 3, 4, 5\}$ . The union set of  $A_1$  and  $A_2$ ,  $\{1, 2, 3, 5\}$  corresponds to the addition of

$A_1 + A_2 = (1\ 1\ 1\ 0\ 1)$  in terms of elementwise Boolean addition.

Boolean addition is defined by  $1 + 1 = 1$ ,  $1 + 0 = 1$ ,  $0 + 1 = 1$  and  $0 + 0 = 0$ . The intersection of  $A_1$  and  $A_2$ , (5) corresponds to the product of  $A_1 \times A_2 = (0\ 0\ 0\ 0\ 1)$  in terms of elementwise Boolean multiplication of 0 and 1. Boolean multiplication follows the rules,  $0 \times 0 = 0$ ,  $1 \times 0 = 0 \times 1 = 0$  and  $1 \times 1 = 1$ . It is clear that these operations satisfy the above relations 1 through 6.

The complement of  $A_k$  is  $A'_k$  whose elements are obtained by switching each element of  $A_k$  to the opposite; thus complement of  $A_1$  is  $(0\ 1\ 0\ 1\ 0)$ ,  $A'_2$  is  $(1\ 0\ 1\ 1\ 0)$  and  $A'_3$  is  $(1\ 0\ 0\ 0\ 0)$ . It is also clear that  $A_k + A'_k$  is equal to 1.

Suppose  $A_k$ ,  $k = 1, \dots, K$  are the row vectors of such a general incidence matrix, and let  $RL_1$  be a set of sets of the attribute vectors. Then  $RL_1$  becomes a sublattice of  $RL$  which is derived from the set of all the response patterns. A subset  $RL_1$  of  $RL$  is called a sublattice if it is closed with respect to the binary compositions  $\cap$  and  $\cup$ . Further Theorem 2 shows that  $RL_1$  becomes a subring of  $RL$  also.

Theorem 2 A set  $RL_1$  of sets of row vectors of an incidence matrix  $Q$  is a Boolean algebra with respect to elementwise Boolean addition and multiplication of 0 and 1.

Boolean addition and multiplication satisfy the following:

- 1  $0 + 0 = 0$
- 2  $1 + 1 = 1$
- 3  $0 + 1 = 1 + 0 = 1$
- 4  $0 \times 0 = 0$
- 5  $0 \times 1 = 1 \times 0 = 0$
- 6  $1 \times 1 = 1$
- 7  $(0 + 1) \times 0 = 1 \times 0 = 0 \text{ \& } (0 \times 1) + 0 = 0 + 0 = 0$
- 8  $(0 + 1) \times 1 = 1 \times 1 = 1 \text{ \& } (0 \times 1) + 1 = 0 + 1 = 1$
- 9  $(0 + 1) + 0 = 0 + (1 + 0) \text{ \& } (0 \times 1) \times 1 = 0 \times (1 \times 1).$

Further  $RL_1$  satisfies  $0' = 1$  and hence  $1' = 0$ . So  $RL_1$  is a Boolean algebra. For any elements of  $RL_1$ ,  $A_k + A_1$  is defined by elementwise Boolean operations of  $+$  and  $\times$ . Then, any elements  $A_k$  and  $A_1$  of  $RL_1$  satisfy the lattice conditions given below:

$$l_1 \quad A_k + A_1 = A_1 + A_k \quad \& \quad A_k \times A_1 = A_1 \times A_k$$

$$l_2 \quad (A_k + A_1) + A_m = A_k + (A_1 + A_m) \quad \&$$

$$A_k \times A_1) \times A_m = A_k \times (A_1 \times A_m)$$

$$l_3 \quad A_k + A_k = A_k \quad \& \quad A_k \times A_k = A_k$$

$$l_4 \quad (A_k + A_1) \times A_k = A_k \quad \& \quad (A_1 \times A_m) + A_1 = A_1$$

Let us define  $\underline{0} = \prod_{k=1}^K A_k$  and  $\underline{1} = \sum_{k=1}^K A_k$ ; then the complement  $A_k'$

is defined by  $A_k + A_k' = \underline{1}$  and  $A_k \times A_k' = \underline{0}$  with elementwise Boolean operations of + and  $\times$ . The distributive laws are also satisfied from properties 7 and 8, that is  $A_k \times (A_1 + A_m) = A_k \times A_1 + A_k \times A_m$ . Therefore  $RL_1$  becomes a Boolean algebra. In the example of our  $3 \times 5$  incidence matrix, the elements  $\underline{0}$  and  $\underline{1}$  are given by  $\underline{0} = (0 \ 0 \ 0 \ 0 \ 1)$  and  $\underline{1} = (1 \ 1 \ 1 \ 1 \ 1)$ .

Example:

$$Q = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\underline{0} = \prod_{k=1}^K A_k = (0 \ 0 \ 0 \ 0 \ 1), \text{ and } \underline{1} = \sum_{k=1}^K A_k = (1 \ 1 \ 1 \ 1 \ 1),$$

$$A_1' = (0 \ 1 \ 0 \ 1 \ 1), A_2' = (1 \ 0 \ 1 \ 1 \ 1), \text{ and } A_3 = (1 \ 0 \ 0 \ 0 \ 1).$$

Several properties of  $RL_1$  are introduced below:

Property 1  $RL_1$  is a subset of all possible response patterns and is closed with respect to the Boolean operations.

Property 2 If  $Q$  is the  $n \times n$  identity matrix, then  $RL_1 = RL$ .

Property 3 If  $A_k \geq A_1$  then  $A_k + A_1 = A_k$  and  $A_k \times A_1 = A_1$ .

Example: Since  $A_3 \geq A_2$ ,  $A_2 + A_3 = A_3$  and  $A_2 \times A_3 = A_2$ .

Property 4 If  $A_k \geq A_1$  then  $A_k' \leq A_1'$ ,  $(A_k + A_1)' = A_k'$  and

$$(A_k \times A_1)' = A_1'.$$

Property 5 If  $Q$  is a  $K \times n$  lower triangle matrix (or Guttman scale matrix) then  $RL_1$  consists of  $K$  row vectors.

If  $Q$  is a Guttman scale matrix, then the row vectors are totally ordered,  $A_1 \leq A_2 \leq \dots \leq A_k$ . For any  $k$  and  $l$ , with  $k \geq l$ ,  $A_k + A_l = A_k$  and  $A_k \times A_l = A_l$  from Property 3. Moreover, the identity  $1$  will be  $A_k$  and the null element  $0$  will be  $A_1$ .

Incidence matrices having this form are often seen in attitude tests where measures are coded by ratings. Models such as Samejima's graded response model or Masters' partial credit model will be suitable to this form of incidence matrices. These models were developed to measure an ordered trait. For such a trait, linearly ordered levels or categories within an item exist.

As a hypothetical example, suppose there are three items:

1) Add  $2/3$  and  $2/3$ , then reduce the answer to its simplest form,

2) Add  $1/3$  and  $1/3$ , and

3) What is the common denominator of  $1/3$  and  $1/5$ ?

Then the attributes are:

$A_1$ : Simplify to the simplest form,

$A_2$ : Get the numerator, and

$A_3$ : Get the denominator.

The incidence matrix is:

$$Q = \begin{matrix} & \text{item 1} & \text{item 2} & \text{item 3} \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} & \begin{matrix} \text{level 1: can do} \\ \text{level 2: can do } A_3 \text{ and } A_2 \\ \text{level 3: can do all} \end{matrix} \end{matrix}$$

Thus, scores for the levels will be 1, 2, and 3, respectively.

Property 6 The complement of the sum of  $A_k$  and  $A_1$  with respect to Boolean addition is the product of the complements of  $A_k$  and  $A_1$ ,  
 $(A_k + A_1)' = A'_k \times A'_1$ .

Property 7 The complement of the product of  $A_k$  and  $A_1$  with respect to Boolean product is the sum of the complements of  $A_k$  and  $A_1$ , i.e.,  $(A_k \times A_1)' = A'_k + A'_1$ .

Definition 6 A chain is a subset of  $RL_1$  in which all the elements are totally ordered with respect to  $\geq$  or  $\leq$ .

Since  $RL_1$  is a partially ordered set, (and so are  $L$ ,  $L_1$  and  $RL$ ) there are usually more than one chain. The order relation is not applicable to two different elements coming from two different chains. Moreover, two chains may contain the same elements in common. Therefore, a tree graph can be drawn by connecting the elements in the chains (Tatsuoka & Tatsuoka, 1990).

The next section introduces a new function by which the universal set of ideal response patterns (or all possible states

of knowledge) is obtainable from an incidence matrix, and gives their descriptive meanings. The description of states are given by a list of combinatorial "can/cannot" attributes.

**Boolean Description Function: Determination of Ideal Response Patterns As Error-Free States of Knowledge and Capabilities**

There are several interesting relationships between the  $K$ -dimensional unit vector  $e_k$  and  $A_k$ .

Property 8 The unit vector  $e_k$  of the latent variable space uniquely corresponds to attribute vector  $A_k$  and the Boolean sum of  $e_k$ ,  $\sum_k e_k$  corresponds to the sum of  $A_k$ ,  $\sum_k A_k$ . Similarly, the Boolean product of the elements of  $e_k$ ,  $\prod_k e_k$  uniquely corresponds to that of  $A_k$ ,  $\prod_k A_k$ .

Since our goal is to draw some inferences about latent score patterns  $Y$  from observable information of item response patterns, it is necessary to introduce a series of hypotheses which convert the latent-but-interpretable information into observable-and-interpretable information. The observable information in our context is obtainable only from item responses and we do not assume observable information from the latent scores.

Definition 7 A hypothesis  $H_k$  is the statement that "one cannot do attribute  $A_k$  correctly but can do the remaining of the attributes."  $H_k$  will produce the item pattern which is the complement of  $A_k$  and represent an interpretable state of

knowledge.

It is clear that if a student cannot perform  $A_k$  correctly but can do the remaining of the attributes right, then the items involving attribute  $A_k$  will have the score of zero but the items not involving  $A_k$  will get the score of 1s. The mapping function,  $A_k \rightarrow A_k'$  that takes the complement is equivalent to applying the hypothesis  $H_k$ .

Property 9 Taking Hypothesis  $H_k$  is equivalent to taking the complement of  $A_k$  and is denoted by  $A'_k$ .

Property 10 The hypothesis  $H_{k_1+\dots+k_l}$  is "one cannot do any of the attributes  $A_{k_1}, A_{k_2}, \dots, A_{k_l}$  correctly but can do the rest of the attributes".

Taking the hypothesis  $H_{k_1+\dots+k_l}$  is equivalent to taking the complement of the addition of  $A_{k_1}, \dots, A_{k_l}$  i.e.,

$$(A_{k_1} + \dots + A_{k_l})' = A_{k_1}' \times \dots \times A_{k_l}'. \quad (12)$$

The item pattern will be

$X_j = 0$  if  $Q_{k_j} = 1$  if there is at least one  $k$  in the set  $\{k_1, k_2, \dots, k_l\}$

$X_j = 1$  if  $Q_{k_j} = 0$  for all  $k$  in the set  $\{k_1, k_2, \dots, k_l\}$ .

As an example, we use the incidence matrix of order  $3 \times 5$  given on p. 11. Table 1 shows various hypotheses and their descriptive

outcomes and resulting ideal item patterns.

Table 1 to be inserted about here

The description of "cannot/can" in the second column of Table 1 corresponds to the latent-score patterns of  $y$ 's given in Table 2. The hypothesis defined in Properties 9 and 10 and Equation (13) provide us with a mapping between attribute patterns and ideal item patterns.

Table 2 to be inserted about here

This mapping is a Boolean function which plays the role of uncovering the contents of a black box. In our situation, latent scores on the attributes become observable via this Boolean function.

Definition 8 The mapping  $f$  from the attribute response space to the item response space is called a Boolean Description Function. The Boolean descriptive function  $f$  satisfies the following property:

Property 11 For Boolean Description Function  $f$  and  $e_k$ ,

$$1 \quad f(e_k') = A_k'$$

$$2 \quad f((e_1 + e_k)') = f(e_1' \times e_k') = A_1' + A_k'$$

$$3 \quad f(0) = \sum_{k=1}^K A_k'$$

$$4 \quad f(I) = I$$

Note that  $\prod_{k=1}^K A_k = 0$ , but  $(\prod_{k=1}^K A_k)' = \sum_{k=1}^K A_k' \neq I$  because  $\sum_{k=1}^K A_k = I$  but  $\sum_{k=1}^K A_k' \neq I$  in  $RL_1$ .

Since  $RL_1$  is a Ring, it is natural to consider the hypotheses that involve interactions of two or more attributes.

Property 12 The hypothesis  $H_{k_1 \times \dots \times k_l}$  is that "one cannot do attributes  $A_{k_1}, \dots, A_{k_l}$  when all of them are involved in a single item but can do each separately and can do the remaining attributes". This hypothesis corresponds to

$$(A_{k_1} \times \dots \times A_{k_l})' = A_{k_1}' + \dots + A_{k_l}'. \quad (14)$$

The item pattern will be

$$X_j = 0 \quad \text{if } Q_{k_1 j} = Q_{k_2 j} = \dots = Q_{k_l j} = 1$$

$$X_j = 1 \quad \text{if there is at least one } k_t \text{ such that } Q_{k_t j} = 0 \text{ for}$$

$k_t, t = 1, 2, \dots, l$ .

The hypotheses of interactions, (14) also produce the ideal item patterns that can be characterized by "can/cannot attributes".

Insert Table 3 about here

In our situation, the latent score patterns of the

attributes become observable via the Boolean description function. If the attribute response space is considered as a linear vector space of  $Y$ , then the ideal item response patterns generated by the hypothesis introduced in Property 10 will be sufficient to describe students' states of knowledge and capabilities. But as can be seen in Table 3,  $RL_1$  contains other ideal item patterns generated by the hypotheses  $H_{k_1 k_2 \dots k_l}$  which involve the interaction of latent score  $Y_s$ . These patterns do not correspond to the latent attribute score patterns  $Y$  in the linear vector space spanned by the  $e_k$ s. For example, the ideal item response pattern corresponding to the interaction of attribute scores  $y_1 \times y_2$  is produced by  $H_{y_1 \times y_2}$ . In other words, the ideal pattern corresponding to the interaction  $y_1 \times y_2$  contains 0s only for the items that involve both the attributes  $A_1$  and  $A_2$ . Since the current test theories such as Item Response Theory models require the assumption of conditional independence of item responses, they may not be applicable to the dataset obtained through the hypothesis of the interaction of scores ( $y_1 y_2$ , or  $X_1 X_2$ ). We will restrict the scope of this study to the linear hypothesis of  $H_{k_1 + \dots + k_l}$ , which requires only linearity of  $y$ .

The Boolean description function  $f$  is not a one-to-one function. As can be seen in Table 1, hypotheses  $H_3$  and  $H_{2+3}$  yield the identical item pattern (1 0 0 0 0), and so do  $H_0$  and  $H_{1+3}$ . The

ideal item patterns resulting from application of two different hypotheses may not be always different, and indeed there is a systematic relation when two hypotheses produce the same ideal item pattern. Property 3, needless to say, implies that any element  $A_1$  smaller than  $A_k$  with respect to the order  $\geq$  in  $L_1$  "degenerates" so that addition of  $A_k$  and  $A_1$  becomes  $A_k$ . That is,  $A_1 + A_k = A_k$  if  $A_k \geq A_1$ . Similarly,  $A_k \times A_1 = A_1$  if  $A_k \geq A_1$ .

A special example that is affected by this degenerative property is the incidence matrix of Guttman type. This type of incidence matrix produces  $K$  elements consisting of the original row vectors because the row vectors become a single chain of length  $K$ . The  $3 \times 5$  incidence matrix used as example above often has two chains,  $A_3$ , and  $A_1 \geq A_2$ . The distinct elements will be  $A_3$  and  $A_1, A_2$ ,

$$A_3 + A_1 = (1 \ 1 \ 1 \ 1 \ 1), \quad A_2 + A_1 = (1 \ 1 \ 1 \ 0 \ 1),$$

$$A_3 \times A_1 = (0 \ 0 \ 1 \ 0 \ 1), \quad A_2 \times A_1 = (0 \ 0 \ 0 \ 0 \ 1).$$

Let us introduce an important definition that will be useful for determining the number of elements in  $RL_1$ .

Definition 9 An element  $A$  of  $L_1$  is an atom if there are no elements between  $A$  and  $0$ , or equivalently if  $A \leq B$  and  $A = B$ , imply  $B = 0$ .

Property 13 Atoms in  $L_1$  can be generated by

$$A_s = \left( \bigcap_{k \in s} A_k \right) \cap \left( \bigcap_{k \notin s} A_k \right)' \text{ for all possible subsets } s \text{ of}$$

$J = \{1, 2, \dots, K\}$ . Or, equivalently  $A_s = \bigcap_{k \in s} A_k \cap \left( \bigcup_{k \in s'} A_k \right)'$ .

$A_s$ s are prospective atoms and some of them may be equal to 0. The intersection of two different atoms is 0:  $A_k \cap A_l = 0$ .

Property 14 Any element  $B$  of  $L_1$  can be written as  $B = \bigcup_{k \in M} A_k$

where  $A_k$  are atoms and  $M$  is an index set.

Examples of Properties 5 and 6 are illustrated with our familiar  $3 \times 5$   $Q$  matrix. Let us consider the index set  $\{1, 2, 3\}$ . Its non-empty subsets are  $\{1\}$ ,  $\{2\}$ ,  $\{3\}$ ,  $\{1, 2\}$ ,  $\{1, 3\}$ ,  $\{2, 3\}$  and  $\{1, 2, 3\}$ . Then

$$a_1 = A_1 \cap (A_2 \cap A_3)' = (1 \ 0 \ 0 \ 0 \ 1)$$

$$a_2 = A_2 \cap (A_1 \cap A_3)' = (0 \ 0 \ 0 \ 0 \ 1)$$

$$a_3 = A_3 \cap (A_1 \cap A_2)' = (0 \ 0 \ 0 \ 1 \ 1)$$

$$a_{12} = A_1 \cap A_2 \cap A_3' = (0 \ 0 \ 0 \ 0 \ 1)$$

$$a_{13} = A_1 \cap A_3 \cap A_2' = (0 \ 0 \ 1 \ 0 \ 1)$$

$$a_{23} = A_2 \cap A_3 \cap A_1' = (0 \ 1 \ 0 \ 0 \ 1)$$

$$a_{123} = A_1 \cap A_2 \cap A_3 = (0 \ 0 \ 0 \ 0 \ 1)$$

As can be seen in the above examples, there are four atoms  $a_1$ ,  $a_3$ ,  $a_{13}$ , and  $a_{23}$  while  $a_2$ ,  $a_{12}$  and  $a_{123}$  are degenerated to the 0 element of  $L_1$ . The original row vectors are written as follows:

$$A_1 = a_1 + a_{13}$$

$$A_2 = a_{23}$$

$$A_3 = a_{23} + a_{13} + a_3.$$

Since every element in  $RL_1$  is expressed by a combination of atoms,

there are  $2^4 = 16$  elements in  $RL_1$ . In general, any element in  $RL_1$  is written by a linear combination of the atoms that are linearly independent. The number of the atoms will determine the number of elements in  $RL$ . The atoms are usually not interpretable unless a test has the identity incidence matrix. The attributes in an identity incidence matrix are atoms.

#### Summary and Discussion

Tatsuoka (1990) discussed an incidence matrix  $Q$  that is an indication matrix of item characteristics with respect to the underlying cognitive processes which are involved in each item. These cognitive tasks are called cognitively relevant attributes in this study. An advantage of expressing the underlying item characteristics explicitly in matrix form is a tremendous benefit: First, it enables us to use a variety of scoring methods such as right or wrong, graded scores, or partial credit scores. Second, it enables us to apply powerful mathematics to investigate systematically a variety of relationships among the unobservable attributes, between the attributes and the items. Third, it enables us to help examine the structure of a test with respect to the underlying cognitive tasks.

Since a set of sets of attributes is a Boolean algebra (Boolean Algebra has been used widely in the theory of combinatorial circuits of electricity and electronics),

unobservable performances on the attributes are viewed as unobservable electric current running through various gates if they are open. An open gate corresponds to an attribute that is answered correctly, and a closed gate to wrong answers. All the gates in a circuit must be open so that the current goes through it. An item can be answered correctly if and only if all the attributes involved in the item can be answered correctly. This is an intuitive analogy between the electricity and electronics and cognitive processes of answering the items, but Boolean Algebra used for explaining various properties of electricity and combinatorial circuits can be applied to explain the underlying cognitive processes of answering the items.

The theoretical foundation of relationships between observable item response patterns and unobservable responses on the attributes which are cognitively relevant to the items is given in this study also. A newly defined Boolean descriptive function  $f$  plays the role of a link between underlying cognitive processes of test items and all the response patterns of these items. Since the model does not expect that responses on the attributes are observable, measures of performances on the attributes can not be obtained directly. However, the Boolean descriptive function converts unobservable states of knowledge and capabilities into a set of observable item patterns which are

called ideal item patterns that are free from measurement errors. The states of knowledge and capabilities are represented by a list of "can/cannot" attributes. The increase of the numbers of states is combinatorial, but Boolean algebra provides us with mathematical tools to overcome the problem of a combinatorial explosion.

Once a list of predetermined groups or states of knowledge and capabilities is determined by a software called "BUGLIB" based on this study, then the notion of "bug distribution" (Tatsuoka and Tatsuoka, 1987; Tatsuoka, 1990) and statistical pattern classification techniques (Tatsuoka, 1985; Lachenbruch, 1975) will enable us to diagnose students' states of knowledge accurately.

Finally, we conclude the study with an important implication for modern test theory. An incidence matrix implicitly indicates that the attribute scores  $\mathbf{y} = (Y_1, Y_2, \dots, Y_K)$  satisfy local independence by a given performance level if we assume local independence at the item level. The Item Response Theory models are built upon this conditional independence of performance level  $\theta$ . However, the Boolean algebra of a set of sets of response patterns is also a Ring, so it permits us to consider the states of knowledge and capabilities derived from the interaction of attribute scores. The Boolean descriptive function generates the ideal item patterns corresponding to the states determined by

using interaction of attributes. Such errors states have been observed in many studies of "bug analysis" (Brown and Burton, 1978; Tatsuoka, 1984). A new model that does not assume local independence will be needed in the future.

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Table 1

Boolean Descriptive Function: Case of Linear Hypothesis

Hypothesis	Interpretation	Ideal Response Pattern
$H_0$	can do everything	(1 1 1 1 1)
$H_1$	cannot $A_1$ , can $A_2, A_3$	(0 1 0 1 0)
$H_2$	cannot $A_2$ , can $A_1, A_3$	(1 0 1 1 0)
$H_3$	cannot $A_3$ , can $A_1, A_2$	(1 0 0 0 0)
$H_{1+2}$	cannot $A_1, A_2$ , can $A_3$	(0 0 0 1 0)
$H_{1+3}$	cannot $A_1, A_3$ , can $A_2$	(0 0 0 0 0)
$H_{2+3}$	cannot $A_2, A_3$ , can $A_1$	(1 0 0 0 0)
$H_{1+2+3}$	cannot $A_1, A_2$ , and $A_3$	(0 0 0 0 0)

Table 2

Correspondence Between Latent Attribute Space and Item Space

Hypothesis	Attribute Score	Item Score
$H_0$	(1 1 1)	(1 1 1 1 1)
$H_1$	(0 1 1)	(0 1 0 1 0)
$H_2$	(1 0 1)	(1 0 1 1 0)
$H_3$	(1 1 0)	(1 0 0 0 0)
$H_{1+2}$	(0 0 1)	(0 0 0 1 0)
$H_{1+3}$	(0 1 0)	(0 0 0 0 0)
$H_{2+3}$	(1 0 0)	(1 0 0 0 0)
$H_{1+2+3}$	(0 0 0)	(0 0 0 0 0)

Table 3

Boolean Descriptive Function: Case of Interaction

Hypothesis	Interpretation	Ideal Response Pattern
$H_{2x3}$	Cannot $A_2$ and $A_3$ together, can $A_1$	(1 0 1 1 0)
$H_{1x3}$	Cannot $A_1$ and $A_3$ together, can $A_2$	(1 1 0 1 0)